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OPTIMISATION OF THE TORSIONAL RIGIDITY OF AXISYMMETRIC HOLLOW SHAFTS

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Abstract—The torsional rigidity of an elastic axisymmetric hollow shaft of prescribed length and internal cross-section is maximised by appeal to the Calculus of Variations. This optimisation is with respect to variation of the outer curved surface of the shaft, subject to the isoperimetric constraint that its volume remains fixed. A regular perturbation solution for the case of small shaft thickness is presented.

1. INTRODUCTION

We consider the maximisation of the torsional rigidity of an axisymmetric hollow, linearly elastic, shaft of prescribed length and internal cross-section, by variation of its outer curved surface under the isoperimetric constraint that its volume remains fixed.

Our approach is similar to that used by Banichuk[1] in the optimisation of cross-sectional shapes of elastic bars in torsion. We cast the governing boundary-value problem in a variational form that readily allows the Calculus of Variations to set a necessary condition for optimality along the outer curved surface. We can then bring numerical methods to the general optimisation problem, and a regular perturbation procedure to the case where the shaft thickness is small in comparison to the internal cross-sectional radius.

2. BOUNDARY-VALUE PROBLEM FOR TORSIONAL RIGIDITY

The shaft of length L is placed with its axis of symmetry along the z-axis, to lie between the planes $z = 0$ and $z = L$. The outer and inner cross-sectional radii are given by $A(z)$ and $a(z)$ respectively. Cylindrical polar coordinates (r, θ, z) are used, the corresponding infinitesimal displacement components in the shaft being denoted by $(u(r, z), v(r, z), w(r, z))$. The sides of the shaft are traction-free, the end $z = 0$ is fixed, and a couple of magnitude M is applied at the end $z = L$ by imposing the boundary conditions

$$v = \alpha r, \qquad \sigma_{zz} = \sigma_{rz} = 0$$

there. Here α is an infinitesimal constant, and σ_{zz} and σ_{rz} denote components of the stress tensor in the usual notation. The torsional rigidity Γ of the shaft is defined as the ratio M/α and it can be evaluated equivalently as

$$\Gamma = M^2/E, \tag{2.1}$$

where E is the total elastic strain energy in the shaft, since E can be identified quickly with αM after an appeal to the divergence theorem and to the prescribed boundary conditions.

It is assumed (in the classical Michell solution) that the displacement components u and w vanish throughout the shaft. Then the introduction of the Michell torsion function by the relations

$$\psi_{,r} = r^3(v/r)_{,z}, \qquad \psi_{,z} = -r^3(v/r)_{,r} \tag{2.2}$$

leaves the equilibrium equations satisfied identically and requires $\psi(r, z)$ to meet the compatibility condition

$$(r^{-3}\psi_{,r})_{,r} + (r^{-3}\psi_{,z})_{,z} = 0 \tag{2.3}$$

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throughout the shaft, where subscripts preceded by a comma denote differentiations. The sides of the shaft are free of traction if

$$\psi = C_1 \quad \text{on} \quad r = a(z), \tag{2.4}$$

$$\psi = C_2 \quad \text{on} \quad r = A(z), \tag{2.5}$$

where C_1 and C_2 are constants that make up the relation

$$M = 2\pi\mu(C_2 - C_1)$$

where μ is the elastic shear modulus. The boundary conditions for the displacement at the ends of the shaft become

$$\psi_{,z} = 0 \quad \text{on} \quad z = 0 \quad \text{and} \quad z = L, \tag{2.6}$$

by appeal to (2.2).

It can be verified readily as an application of the extremum principle of minimum potential energy that the solution ψ coincides with the particular function ψ that minimises the functional $E[\psi, A]$ defined by

$$E[\psi, A] = \frac{1}{2}\pi\mu \int_0^L dz \int_{a(z)}^{A(z)} r^{-3}(\psi_{,r}^2 + \psi_{,z}^2)\, dr,$$

where the minimisation is with respect to all continuously differentiable functions ψ that meet the constraints (2.4)–(2.6). The minimising function ψ necessarily satisfies also the compatibility condition (2.3), and the minimum value of $E[\psi, A]$ is identical to the strain energy E introduced in (2.1) to evaluate Γ.

3. OPTIMALITY CONDITION

The external cross-sectional radius $A(z)$ is now allowed to vary subject to the isoperimetric constraint

$$\pi \int_0^L (A^2 - a^2)\, dz = V_0 \tag{3.1}$$

which keeps the volume of material in the shaft fixed at the value V_0. Then the torsional rigidity Γ will be taken to its desired maximum, by the optimal external radius $A_0(z)$, say, with associated torsion function $\psi_0(z)$, at the same time as the functional $E[\psi, A]$ is minimised with respect to variations of ψ and A that obey the constraints (2.4)–(2.6) and (3.1). Therefore a necessary optimality requirement is disclosed in the amount ΔE by which $E[\psi, A]$ exceeds the minimum value $E[\psi_0, A_0]$ under the weak variation

$$\psi = \psi_0 + \epsilon\psi_1 + 0(\epsilon), \qquad A = A_0 + \epsilon A_1 + 0(\epsilon),$$

where ϵ is a small positive parameter, and where the constraints imposed on ψ and A and equally on ψ_0 and A_0 restrict the variations ψ_1 and A_1. It should be emphasised that ψ_1 is not necessarily forced to vanish at the optimal radius A_0 (as the analysis of Banichuk[1] at a corresponding stage would seem to presume, without however vitiating his particular sequel). Rather, the constraint (2.5) when expanded for small ϵ demands that

$$\psi_1 = -A_1\psi_{0,r} \quad \text{on} \quad r = A_0(z).$$

At the same time (3.1) requires A_1 to be such that

$$\int_0^L A_0 A_1\, dz = 0. \tag{3.2}$$

We are now able to calculate that

$$\Delta E = -\epsilon\pi \int_0^L A_1 A_0^{-3}[\psi_{0,n}(A_0(z),z)]^2 \,dz$$
$$-2\epsilon\pi \int_{S_0} \psi_1[(r^{-3}\psi_{0,r})_{,r} + (r^{-3}\psi_{0,z})_{,z}]\,dS + 0(\epsilon)$$

where S_0 denotes the optimal plane region for which θ is a constant and $a(z) \leq r \leq A_0(z)$, $0 \leq z \leq L$, and where the function $\psi_{0,n}$ is defined as $(\psi_{0,r}^2 + \psi_{0,z}^2)^{1/2}$. Because of the necessary vanishing here of the coefficient of ϵ, we recover first of all the compatibility condition (2.3) in S_0 and secondly in view of (3.2) we deduce (again by standard arguments) as a necessary condition for the optimality of S_0 that

$$A_0^{-2}\psi_{0,n}(A_0(z), z) = \text{constant}, \tag{3.3}$$

in other words that over the optimal surface $r = A_0(z)$ the normal derivative of the torsion function ψ_0 varies as the square of A_0. It is interesting to observe that the optimal surface is consequently one on which a constant value is taken by the "stress magnitude" τ given by

$$\tau = (\sigma_{r\theta}^2 + \sigma_{z\theta}^2)^{1/2} = \mu A_0^{-2}(\psi_{0,r}^2 + \psi_{0,z}^2)^{1/2},$$

and so according to Wheeler[2] it is one which also minimises the magnitude of the maximum elastic stress concentration in the shaft.

When the boundary-value problem of Section 2 is set in the two-dimensional region S_0, it is presumed that the extra boundary condition (3.3) together with the isoperimetric condition (3.1) enables the determination of the optimal curve $r = A_0(z)$ giving maximum torsional rigidity for prescribed V_0, L and $a(z)$. However when all these necessary conditions are satisfied it must admittedly remain difficult to verify with mathematical certainty that the ultimate maximum has been indeed attained; we may appeal to the general analysis of Prager[3] for some support. The most immediate (and possibly the most important) solution is that for a constant $a(z)$ in which case the solution is the familiar one

$$\psi_0 = \lambda_0 r^4, \qquad A_0 = b,$$

of a hollow circular cylinder, where the constants λ_0 and b are given by the relations

$$M = 2\pi\mu\,\lambda_0(b^4 - a^4), \qquad V_0 = \pi L(b^2 - a^2).$$

In the general case where $a(z)$ varies arbitrarily along the shaft an analytical solution does not seem forthcoming. Instead we have adapted the numerical technique for free-boundary problems described by Allen[4]. Standard successive overrelaxation methods are employed in an iterative solution procedure. However the accuracy is found to suffer in the end from our lack of an efficient numerical algorithm for making the choices of trial curves. We proceed therefore to a case of small shaft thickness which succumbs to an analytic perturbation procedure (that could be applied equally well to the case where the outer cross-section is prescribed while the inner one is to be optimised).

4. SMALL SHAFT THICKNESS

We consider the case of small shaft thickness for which the ratio V_0/L^3 is prescribed small compared to unity and for which $a(z)$ is of the order of L and is infinitely differentiable with nth derivative of order L^{-n+1} along the entire length of the shaft, and is moreover (for convenience of calculation) such that

$$a'(0) = a'(L) = 0, \tag{4.1}$$

where as usual the prime denotes a first derivative. It is convenient next to recast the boundary-value problem for the optimal region S_0 in a non-dimensional form by the substitutions

$$z = L\bar{z}, \quad r = L\bar{r}, \quad a = L\bar{a}, \quad A_0 = L\bar{A},$$
$$M = \bar{\epsilon}\mu L^3\bar{M}, \quad C_1 = \bar{\epsilon}L^3\bar{C}_1, \quad C_2 = \bar{\epsilon}L^3\bar{C}_2,$$
$$\psi_0 = \bar{\epsilon}L^3\bar{\psi}, \quad \bar{\epsilon} = V_0/L^3,$$

in favour of "barred" variables (which need no longer carry the subscript 0). Furthermore, it is helpful to proceed at once to a change of coordinates by means of the transformation

$$\bar{r} = \bar{a}(\bar{z}) + \bar{\epsilon}Y, \quad \bar{z} = t,$$

after which the region S_0 in the $r-z$ plane is taken into the region R in the $t-Y$ plane specified by

$$0 \le Y \le h(t), \quad 0 \le t \le 1,$$

say. The constant $\bar{C}_1$ can be chosen to vanish without loss of generality, while its companion $\bar{C}_2$ can be replaced henceforth by k $(= \bar{M}/2\pi)$. Finally therefore, we find that there remains the problem of determining the functions $\bar{\psi}(t, Y)$ and $h(t)$, together with a constant λ, brought in by the condition (3.3), subject to the requirements that

$$(1+\bar{a}'^2)\bar{\psi},_{YY} - 2\bar{a}'\bar{\epsilon}\bar{\psi},_{Yt} - \bar{a}''\bar{\epsilon}\bar{\psi},_Y + \bar{\epsilon}^2\bar{\psi},_{tt} - 3(\bar{a}+\bar{\epsilon}Y)^{-1}\bar{\epsilon}\bar{\psi},_Y = 0 \quad \text{in} \quad R,$$
$$\bar{\psi} = 0 \quad \text{on} \quad Y = 0, \qquad \bar{\psi} = k \quad \text{on} \quad Y = h,$$
$$\bar{\psi},_t = 0 \quad \text{at} \quad t = 0 \quad \text{and} \quad t = 1,$$
$$(\bar{a}+\bar{\epsilon}h)^{-2}[1+(\bar{a}'+\bar{\epsilon}h')^2]^{1/2}\bar{\psi},_Y = \lambda \quad \text{on} \quad Y = h,$$
$$\pi\int_0^1 (2\bar{a}h + \bar{\epsilon}h^2)\,\mathrm{d}t = 1, \tag{4.2}$$

that is, to the governing differential equation, the boundary and isoperimetric conditions.

The problem permits the regular perturbation solution

$$\bar{\psi} = \bar{\psi}^0 + \bar{\epsilon}\bar{\psi}^1 + \dots, \quad h = h^0 + \bar{\epsilon}h^1 + \dots, \quad \lambda = \lambda^0 + \bar{\epsilon}\lambda^1 + \dots,$$

after the various bracketed quantities in (4.2) are expanded likewise as power series in $\bar{\epsilon}$ (where $0 < \bar{\epsilon} \ll 1$). The solution to the order zero (and hence to the higher orders) is found easily in the present circumstances where the condition (4.1) applies, that is where $\bar{a}'(t)$ vanishes at $t = 0$ and $t = 1$. Thus it is found that

$$\bar{\psi}^0 = \lambda^0\bar{a}^2(1+\bar{a}'^2)^{-(1/2)}Y, \quad h^0 = k(1+\bar{a}'^2)^{1/2}/\lambda^0\bar{a}^2,$$

where

$$\lambda^0 = 2\pi k\int_0^1 \bar{a}^{-1}(1+\bar{a}'^2)^{1/2}\,\mathrm{d}t.$$

Put back in terms of the original dimensional variables, the zero-order approximations to the optimal, outer cross-sectional radius $A_0(z)$ and torsional rigidity Γ are given in terms of the prescribed length L, volume V_0 and inner radius $a(z)$ by

$$A_0(z) - a(z) = V_0(1+\{a'(z)\}^2)^{1/2}/2\pi K\{a(z)\}^2, \quad \Gamma = 2\mu V_0/K^2,$$

where

$$K = \int_0^L (1 + \{a'(z)\}^2)^{1/2}\{a(z)\}^{-1}\,dz \qquad (= \mu V_0 \lambda^0/M),$$

while the accompanying approximation to the torsion function is

$$\psi_0 = MK\{a(z)\}^2(1 + \{a'(z)\}^2)^{-(1/2)}\{r - a(z)\}/\mu V_0.$$

We can observe here the extent to which the optimal difference $A_0(z) - a(z)$ is increased or decreased at points where $a'(z)$ or $a(z)$, respectively, is prescribed relatively large.

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